

A bound for the number of lines lying on a nonsingular surface in 3-space over a finite field

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Abstract

A nonsingular surface of degree $d \geq 2$ in \mathbb{P}^3 over \mathbb{F}_q has at most $((d-1)q+1)d$ \mathbb{F}_q -lines, and this bound is optimal for $d = 2, \sqrt{q} + 1, q + 1$. This is a bi-product of a previous study on estimating the number of \mathbb{F}_q -points of surfaces.

Key Words: Finite field, Surface, Number of lines

MSC: 14G15, 14J70, 14N05, 14N10

This short note is a supplement to our previous works [2, 3, 4]. In [3, 4], we considered the set of \mathbb{F}_q -lines on a given surface $S \subset \mathbb{P}^3$ as an auxiliary tool in order to estimate the number of \mathbb{F}_q -points of S . This time, we turn our attention to \mathbb{F}_q -lines themselves on S .

Setting 1 Let S be a nonsingular surface of degree $d \geq 2$ in \mathbb{P}^3 defined over \mathbb{F}_q . Let $\nu_q(S)$ denote the number of \mathbb{F}_q -lines lying on S , and $N_q(S)$ that of \mathbb{F}_q -points of S .

In [2, 4], we showed the following fact, in which $N_q(X)$ denotes the number of \mathbb{F}_q -points of the surface X , though X is not necessary nonsingular.

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Theorem 2 *Let X be a surface in \mathbb{P}^3 over \mathbb{F}_q without \mathbb{F}_q -plane components, then*

$$N_q(X) \leq ((d-1)q+1)(q+1). \quad (1)$$

Furthermore, equality holds in (1), if and only if the surface X is projectively equivalent to one of the following surfaces over \mathbb{F}_q :

- (i) $X_0X_1 - X_2X_3 = 0$ if $d = 2$;
- (ii) $X_0^{\sqrt{q}+1} + X_1^{\sqrt{q}+1} + X_2^{\sqrt{q}+1} + X_3^{\sqrt{q}+1} = 0$ if $d = \sqrt{q} + 1$;
- (iii) $X_0X_1^q - X_0^qX_1 + X_2X_3^q - X_2^qX_3 = 0$ if $d = q + 1$.

In this note, we shall give a proof of the following theorem. In the proof, $\#T$ denotes the cardinality of T if T is a finite set.

Theorem 3 *Under Setting 1,*

$$\nu_q(S) \leq \frac{d}{q+1}N_q(S) \leq ((d-1)q+1)d. \quad (2)$$

Furthermore the list of surfaces S satisfying $\nu_q(S) = ((d-1)q+1)d$ coincides with that in Theorem 2.

Proof. Let $G(1, \mathbb{P}^3)$ be the Grassmann variety of lines in \mathbb{P}^3 . The sets of \mathbb{F}_q -points of S and $G(1, \mathbb{P}^3)$ are denoted by $S(\mathbb{F}_q)$ and $G(1, \mathbb{P}^3)(\mathbb{F}_q)$ respectively. Consider the correspondence

$$\Pi = \{(P, l) \mid P \in l \subset S\} \subset S(\mathbb{F}_q) \times G(1, \mathbb{P}^3)(\mathbb{F}_q)$$

with projections $\pi_1 : \Pi \rightarrow S(\mathbb{F}_q)$ and $\pi_2 : \Pi \rightarrow G(1, \mathbb{P}^3)(\mathbb{F}_q)$. Then $\#(\text{Im } \pi_2) = \nu_q(S)$, and hence $\#\Pi = (q+1)\nu_q(S)$. On the other hand, a line l on S passing through P lies on the tangent plane $T_P(S)$ to S at P because $l = T_P(l) \subset T_P(S)$. Hence the number of lines on S passing through the assigned point P is at most that of line components of the curve $S \cap T_P(S)$. Hence $\#\pi_1^{-1}(P) \leq d$. So, to sum up, $(q+1)\nu_q(S) = \#\Pi \leq N_q(S)d$, which is the first inequality in (2). The second one comes from Theorem 2.

Next we show the additional statement. If $\nu_q(S) = ((d-1)q+1)d$, then $N_q(S) = ((d-1)q+1)(q+1)$ by (2), and hence S is one of the surfaces listed in Theorem 2. Conversely we show that the following claim holds for each surface S defined by (i) or (ii) or (iii);

Claim: for any $P \in S(\mathbb{F}_q)$, the intersection of the tangent plane $T_P(S)$ with S is a union of d \mathbb{F}_q -lines with vertex P , that is, $T_P(S) \cap S$ forms a planar pencil with vertex P in terms of [4]. This claim actually implies $\nu_q(S) = ((d-1)q+1)d$. Indeed, $\#\pi_1^{-1}(P) = d$ in the first part of this proof, together with $N_q(S) = ((d-1)q+1)(q+1)$, gives rise to

$$(q+1)\nu_q(S) = \#\Pi = N_q(S)d = ((d-1)q+1)(q+1)d.$$

This claim had been proved essentially in [4, Prop. 3.1], however, here we give a direct proof by using equations (i), (ii) or (iii). Let $P = (a_0, \dots, a_3) \in S(\mathbb{F}_q)$.

(i) Suppose that S is defined by equation (i). Then the tangent plane $T_P(S)$ is defined by

$$a_1X_0 + a_0X_1 - a_3X_2 - a_2X_3 = 0.$$

Without loss of generality, we may assume that $a_1 = 1$, hence $a_0 = a_2a_3$. Then $S(\mathbb{F}_q) \cap T_P(S)$ is defined by

$$\begin{aligned} 0 &= (-a_2a_3X_1 + a_3X_2 + a_2X_3)X_1 - X_2X_3 \\ &= (a_2X_1 - X_2)(X_3 - a_3X_1) \end{aligned}$$

in $T_P(S) = \mathbb{P}^2$ with coordinates X_1, X_2, X_3 .

(ii) Suppose that S is defined by equation (ii). Since any automorphism of S comes from an \mathbb{F}_q -linear transformation of \mathbb{P}^2 and the automorphism group acts on $S(\mathbb{F}_q)$ as transitively [1, Lemmas 3.7 and 3.8], we may assume that $P = (1, \zeta, 0, 0)$ with $\zeta^{\sqrt{q}+1} = -1$. Hence $T_P(S)$ is defined by $X_0 + \zeta^{\sqrt{q}}X_1 = 0$, and $T_P(S) \cap S$ by $X_2^{\sqrt{q}+1} + X_3^{\sqrt{q}+1} = 0$ in $T_P(S) = \mathbb{P}^2$ with coordinates X_1, X_2, X_3 . Since

$$X_2^{\sqrt{q}+1} + X_3^{\sqrt{q}+1} = \prod_{\substack{\lambda \in \mathbb{F}_q^* \\ \text{with } Nm \lambda = 1}} (X_2 - \lambda \zeta X_3),$$

$T_P(S) \cap S$ splits into $\sqrt{q} + 1$ lines.

(iii) Suppose that S is defined by equation (iii). Since $S(\mathbb{F}_q) = \mathbb{P}^3(\mathbb{F}_q)$, it is enough to show that for $Q = (b_0, \dots, b_3) \in S(\mathbb{F}_q) \cap T_P(S)$, the line $PQ = \{\lambda(a_0, \dots, a_3) + \mu(b_0, \dots, b_3) \mid (\lambda, \mu) \in \mathbb{P}^1\}$ joining P and Q lies on S . Since $P = (a_0, \dots, a_3)$ is an \mathbb{F}_q -point, the tangent plane $T_P(S)$ is defined by

$$a_1X_0 - a_0X_1 + a_3X_2 - a_2X_3 = 0.$$

Hence

$$a_1b_0 - a_0b_1 + a_3b_2 - a_2b_3 = 0.$$

Therefore

$$\begin{aligned} &(\lambda a_0 + \mu b_0)(\lambda a_1 + \mu b_1)^q - (\lambda a_0 + \mu b_0)^q(\lambda a_1 + \mu b_1) \\ &\quad + (\lambda a_2 + \mu b_2)(\lambda a_3 + \mu b_3)^q - (\lambda a_2 + \mu b_2)^q(\lambda a_3 + \mu b_3) \\ &= \lambda^q \mu (a_1b_0 - a_0b_1 + a_3b_2 - a_2b_3) + \lambda \mu^q (a_0b_1 - a_1b_0 + a_2b_3 - a_3b_2) \end{aligned}$$

is identically 0. □

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